# NDIM achievements: Massive, Arbitrary tensor rank and N-loop insertions in Feynman integrals

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One of the main difficulties in studying Quantum Field Theory, in the perturbative regime, is the calculation of D-dimensional Feynman integrals. In general, one introduces the so-called Feynman parameters and associated with them the cumbersome parametric integrals. Solving these integrals beyond the one-loop level can be a difficult task. Negative dimensional integration method (NDIM) is a technique whereby such problem is dramatically reduced. In this work we present the calculation of two-loop integrals in three diferent cases: scalar ones with three diferent masses, massless with arbitrary tensor rank, with N-insertions of a 2-loop diagram.

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## I. INTRODUCTION.

Perturbative calculations in Quantum Field Theory are often a hard task, specially if one does not have a suitable approach to tackle the problem. Among the several techniques available in the market the most popular is the Feynman parametrization [1]; in fact, if one is clever enough hefty calculations (at 2-loop level) can be done [2]. However, in our point of view, this is not the most adequate nor elegant method to solve Feynman integrals whether one is considering covariant or non-covariant gauges.

On the other hand, Chetyrkin *et al* developed the integration by parts in configuration space (associated with Gegenbauer polynomials) and performed (at 4-loops) even heftier calculations [3]. However, their technique has a drawback: if the diagrams have more than two external legs manipulation of Gegenbauer polynomials becomes very difficult [2] to handle.

Mellin-Barnes' contour integration is a third option in the market. Each propagator is Mellin-transformed [4] – a very simple step – and using Barnes' lemmas and summing over the residues, it is possible to write the Feynman integrals as hypergeometric functions or hypergeometric-like series. Smirnov [5] solved the massless double-box with Mellin-Barnes approach. Such Mellin integrals are parametric-like integrals – though they are much simpler to solve than Feynman-like ones because Cauchy theorem can be applied straightforwardly.

NDIM [6] is a technique where the parametric integrals do not appear. We start up with a Gaussian integral, which is well-behaved, perform Taylor expansion and solve systems of algebraic equations. All the calculations can be done analytically and the results are given for arbitrary exponents of propagators and space-time dimension $D$, as in the standard dimensional regularization [7]. Even integrals pertaining to the trickier non-covariant gauges, such the light-cone [8] and Coulomb [9] ones, can be performed using the same approach (however, we will deal with them in another work). In the following sections we will show how this can be done.

The outline for our paper is as follows: in section II we consider the simplest two-loop diagram, our workhorse, as a pedagogical example and to present, in a clean way, the technique of negative-dimensional integration. Section III is devoted to the referred diagram but now with tensorial structure. The NDIM can handle vector, second rank tensor and higher integrals, all at the same time. In the fourth section we replace massless propagators by massive ones and in section V, we consider the massless diagram with N-insertions of the same type. In the last section, VI, we present our concluding remarks.

## II. SIMPLEST TWO-LOOP DIAGRAM.

To make things clear we begin with the diagram of figure 1. In a massless scalar theory it is represented by,

$$A = \int \frac{d^Dq\; d^Dk}{(q^2)(k^2)(p-q-k)^2}. \tag{1}$$

*Negative-dimensional approach.*

Consider the gaussian integral,

$$G=\int d^Dq\; d^Dk \;\; \exp\left[-\alpha q^2-\beta k^2-\gamma(p-q-k)^2\right], \tag{2}$$

where $(\alpha,\beta,\gamma)$ are such that $G$ is well-behaved. We will see that it is the generating functional of negative-dimensional integrals, see eq.(5). Integrating over $q$ and $k$ is very easy,

$$G=\left(\frac{\pi^2}{\lambda}\right)^{D/2}\exp\left(-\frac{\alpha\beta\gamma}{\lambda}p^2\right), \tag{3}$$

where $\lambda=\alpha\beta+\alpha\gamma+\beta\gamma$. Expanding (3) in Taylor series and using a multinomial expansion for $\lambda$ we get,

$$G=\pi^D\sum_{n_1,n_2,n_3,n_4=0}^{\infty}\frac{\alpha^{n_{123}}\beta^{n_{124}}\gamma^{n_{134}}(p^2)^{n_1}}{n_1!n_2!n_3!n_4!}(-n_1-D/2)!, \tag{4}$$

where due to our multinomial expansion, $n_{234}=-n_1-D/2$, and we define

$$n_{12}=n_1+n_2, \qquad\qquad n_{123}=n_1+n_2+n_3,$$

and so forth.

On the other hand, Taylor expanding (2),

$$G=\sum_{i,j,l=0}^{\infty}\frac{(-1)^{i+j+l}\alpha^i\beta^j\gamma^l}{i!j!l!}\int d^Dq\; d^Dk\;\; (q^2)^i(k^2)^j(p-q-k)^{2l}, \tag{5}$$

one generates the negative-$D$ integral. Now, comparing (4) and (5) we solve for the integral above,

$$\mathcal{A}(i,j,l)=\int d^Dq\; d^Dk\;\; (q^2)^i(k^2)^j(p-q-k)^{2l} \tag{6}$$

$$=\frac{\pi^D i!j!l!}{(-1)^{i+j+l}}\sum_{n_1,n_2,n_3,n_4=0}^{\infty}\frac{(p^2)^{n_1}(-n_{1234}-D/2)!}{n_1!n_2!n_3!n_4!}\delta_{n_{123},i}\delta_{n_{124},j}\delta_{n_{134},l}\delta_{n_{234},-n_1-D/2}, \tag{7}$$

where the Kronecker deltas give rise to a system of 4 equations and 4 “unknowns”. Plugging the solution into (7) provides us the result of $\mathcal{A}(i,j,l)$, namely,

$$\mathcal{A}(i,j,l)=\frac{\pi^D\Gamma(1+i)\Gamma(1+j)\Gamma(1+l)\Gamma(1-\sigma-D/2)}{\Gamma(1-i-D/2)\Gamma(1-j-D/2)\Gamma(1-l-D/2)\Gamma(1+\sigma)}(p^2)^\sigma, \tag{8}$$

where $\sigma=i+j+l+D$. However, the above result is valid on the negative-dimensional region and positive exponents of propagators $(i,j,l\geq 0)$. To bring it to our real physical world we must invoke the principle of analytic continuation. This is a quite simple step [10]: group the gamma functions into Pochhammer symbols and use one of its properties,

$$(a)_k\equiv(a|k)=\frac{\Gamma(a+k)}{\Gamma(k)}, \qquad\qquad (a|-k)=\frac{(-1)^k}{(1-a|k)}. \tag{9}$$

Doing so, we get the result in our positive-dimensional world,

$$\mathcal{A}^{AC}(i,j,l)=\pi^D(p^2)^\sigma(-i|i+j+D/2)(-j|j+k+D/2)(-k|i+k+D/2)(\sigma+D/2|-2\sigma-D/2), \tag{10}$$

and negative exponents of propagators $(i,j,l)\leq 0$ in Euclidean space. Observe that the result is symmetric in the propagators exponents reflecting the symmetry of figure 1.

The recipe for calculating Feynman integrals using NDIM technology is quite simple: i) to each loop write a Gaussian integral whose arguments are the propagators of the diagram in question; ii) complete the square(s) and integrate; iii) take the original Gaussian integral, Taylor expand the exponential and change the order $\sum\leftrightarrow\int$; this operation generates negative-dimensional integrals; iv) the equality of these two expressions must hold, so one can solve for the negative-dimensional integral and gets $n$-fold series involving Kronecker deltas; v) such Kronecker deltas give rise to a system of linear algebraic equations, in most cases it has not a unique solution, since it is a rectangular matrix [11]; vi) Plugging the solution(s) into that series representation provides the result of negative-dimensional integral – sometimes in massless cases there appears degenerate solutions [10]; vii) analytically continue the referred result to positive-dimensional region using the above property of Pochhammer symbols. The whole procedure is quite simple, and we will show that in cases of interest.

## III. TENSORIAL STRUCTURE.

The previous result was obtained with amazing easiness: arbitrary negative exponents of propagators, positive dimension and no numerical calculations. The reader can rightfully ask: Does NDIM work for tensorial numerators as well? The answer is yes [12]. We need to modify only one thing.

Consider the integral,

$$\mathcal{B}(i,j,l,m) = \int d^Dq\, d^Dk\ \ (q^2)^i (k^2)^j (p-q-k)^{2l} (2q\cdot p)^m. \tag{11}$$

From,

$$\begin{aligned} G_T &= \int d^Dq\, d^Dk\ \ \exp\left[-\alpha q^2 - \beta k^2 - \gamma (p-q-k)^2 - 2\phi q\cdot p\right] && (12)\\ &= \left(\frac{\pi^2}{\lambda}\right)^{D/2} \exp\left(-\frac{\alpha\beta\gamma + 2\beta\gamma\phi - \beta\phi^2 - \gamma\phi^2}{\lambda} p^2\right), \\ &= \pi^D \sum_{n_1,\ldots,n_7=0}^{\infty} \frac{(-1)^{n_{12}} 2^{n_2} \alpha^{n_{157}} \beta^{n_{12356}} \gamma^{n_{12467}} \phi^{n_2+2n_{34}} (p^2)^{n_{1234}}}{n_1!n_2!n_3!n_4!n_5!n_6!n_7!} (-n_{1234} - D/2)!, && (13)\\ &= \sum_{i,j,l,m=0}^{\infty} \frac{(-1)^{i+j+l+m} \alpha^i \beta^j \gamma^l \phi^m}{i!j!l!m!} \int d^Dq\, ; d^Dk\ (q^2)^i (k^2)^j (p-q-k)^{2l} (2q\cdot p)^m. && (14) \end{aligned}$$

As we did in the previous section, solving for $\mathcal{B}$, leads to

$$\begin{aligned} \mathcal{B}(i,j,l,m) = \frac{\pi^D i!j!l!m!}{(-1)^{i+j+l+m}} &\sum_{n_1,\ldots,n_7=0}^{\infty} \frac{2^{n_2}(-1)^{n_{12}}(p^2)^{n_{1234}}(-n_{1234}-D/2)!}{n_1!n_2!n_3!n_4!n_5!n_6!n_7!} \delta_{n_{157},i}\delta_{n_{12356},j}\delta_{n_{12467},l} \\ &\times \delta_{n_{n_2+2n_{34}},m}\delta_{n_{567},-n_{1234}-D/2}. \end{aligned} \tag{15}$$

Observe that now the system generated by the Kronecker deltas does not have a unique solution, since there are seven "unknowns" and five equations and, by this very reason, two of them will be left undetermined. There are ($C^7_5 = 7!/5!2! = 21$) distinct ways of solving this $5\times 7$ system, but 5 of them have no solution at all.

In a previous work [12] we did show that all non-trivial solutions are legitimate and lead to the same result for the Feynman integral in question. Here the same occurs – but we will not prove it.

Note that the tensorial sector of the Feynman integral $\mathcal{B}(i,j,l,m)$ is contained in the factor $(2q\cdot p)^m$ and for this very reason the exponent $m$ can not be analytically continued to allow for negative values. In other words, we must invoke the principle of analytic continuation for three exponents of propagators leaving the fourth, $m$, untouched.

All the solutions will give rise to a double series of hypergeometric type, since we have a 7-fold series and only 5 Kronecker deltas in (15). However, from the theory of hypergeometric series [13] we know that when one of its numerator parameters is a negative integer, say $-m$, the series is truncated and has only $m$ terms. For this reason we will consider a solution that is obtained when $\{n_3, n_4\}$ are left undetermined.

$$\mathcal{B}(i,j,l,m) = g_{\mathcal{B}} \sum_{n_3,n_4=0}^{\infty} \frac{(-m/2|n_{34})(1/2-m/2|n_{34})(D/2+j|n_4)(D/2+l|n_3)}{(1+\sigma'-m|n_{34})(1-i-m-D/2|n_{34})n_3!n_4!}, \tag{16}$$

where

$$g_{\mathcal{B}} = \frac{(-\pi)^D (p^2)^{\sigma'} 2^m \Gamma(1+i)\Gamma(1+j)\Gamma(1+l)\Gamma(1-\sigma'-D/2)}{\Gamma(1-j-D/2)\Gamma(1-l-D/2)\Gamma(1-i-m-D/2)\Gamma(1+\sigma'-m)}, \tag{17}$$

$\sigma' = \sigma + m$ and we use the relation [13],

$$(a|2b) = 2^{2b}(a/2|b)(1/2+a/2|b).$$

Note that for positive $m$, which is the relevant condition, the series (16) is always truncated (when it is even the first factor in the numerator is the responsible for that, whereas for $m$ odd the second factor truncates the series.) Analytically continuation of $g_{\mathcal{B}}$ gives,

$$g_{\mathcal{B}}^{AC}=\pi^D(p^2)^{\sigma'}2^m(-i|-j-l-D)(-j|j+l+D/2)(-l|j+l+D/2)(\sigma'+D/2|i+m-\sigma'), \tag{18}$$

and the final result, in positive dimension, is given by the series in equation (16) times $g_{\mathcal{B}}^{AC}$,

$$\mathcal{B}^{AC}(i,j,l,m)=g_{\mathcal{B}}^{AC}\sum_{n_3,n_4=0}^{\infty}\frac{(-m/2|n_{34})(1/2-m/2|n_{34})(D/2+j|n_4)(D/2+l|n_3)}{(1+\sigma'-m|n_{34})(1-i-m-D/2|n_{34})n_3!n_4!}. \tag{19}$$

Observe that for $m=0$ we obtain the scalar case (10), for $m=1$ an integral with vector numerator, for $m=2$ second rank tensor and so forth. The results, of course, are contracted with the external momentum $p^\mu$. The astonishing point is that all these **new** results are contained *in the same formula*, namely equation (19).

## IV. MASSIVE PROPAGATORS.

NDIM is a powerful technique. It gives, simultaneously, vector, second rank tensor and higher order integrals. A second question one could ask is: Does NDIM work for massive propagators as well? The answer is also yes and we need to do only slight modifications.

Let our generating function, corresponding to diagram of figure 1 where now the virtual particles have distinct masses, be

$$\begin{aligned} G_m &= \int d^Dq\; d^Dr\;\; \exp\left\{-\alpha(q^2-m_1^2)-\beta(r^2-m_2^2)-\gamma\left[(p-q-r)^2-m_3^2\right]\right\} \\ &= \sum_{i,j,k=0}^{\infty}\frac{(-1)^{i+j+k}\alpha^i\beta^j\gamma^k}{i!j!k!}\mathcal{M}(i,j,k) \end{aligned} \tag{20}$$

where,

$$\mathcal{M}(i,j,k)=\int d^Dq\; d^Dr\;\; (q^2-m_1^2)^i(r^2-m_2^2)^j\left[(p-q-r)^2-m_3^2\right]^k, \tag{21}$$

using (3) we get,

$$G_m=\exp\left(\alpha m_1^2+\beta m_2^2+\gamma m_3^2\right)G, \tag{22}$$

and following the general procedure, as in the previous cases, one can write the integral as,

$$\mathcal{M}(i,j,k)=\frac{\pi^D i!j!k!}{(-1)^{i+j+k}}\sum_{n_1,\ldots,n_7=0}^{\infty}\frac{(-n_4-D/2)!(m_1^2)^{n_1}(m_2^2)^{n_2}(m_3^2)^{n_3}(-p^2)^{n_4}}{n_1!\ldots n_7!}\delta_{n_{1456},i}\delta_{n_{2457},j}\delta_{n_{3467},k}\delta_{n_{4567},-D/2}. \tag{23}$$

In this case, the Kronecker deltas give rise to a $4\times 7$ system of linear algebraic equations. We have 35 possible solutions for such system but 15 of them have no solution at all. So, we are left with 20 triple series, following the prescription of summing the ones which have the same variables [11,8] (it is equivalent to say that we sum the ones which have the same region of convergence [10]), we get four possible triple series (of hypergeometric type) representing the Feynman integral $\mathcal{M}(i,j,k)$,

$$\left(\frac{m_1^2}{p^2}\right)^a\left(\frac{m_2^2}{p^2}\right)^b\left(\frac{m_3^2}{p^2}\right)^c,\left(\frac{m_1^2}{m_3^2}\right)^a\left(\frac{m_2^2}{m_3^2}\right)^b\left(\frac{p^2}{m_3^2}\right)^c,\left(\frac{m_1^2}{m_2^2}\right)^a\left(\frac{m_3^2}{m_2^2}\right)^b\left(\frac{p^2}{m_2^2}\right)^c,\left(\frac{m_2^2}{m_1^2}\right)^a\left(\frac{m_3^2}{m_1^2}\right)^b\left(\frac{p^2}{m_1^2}\right)^c.$$

For the first one we have eight solutions, the second, third and fourth have four, so we have 20 possible solutions of the system generated by the Kronecker deltas $(8+4+4+4=20)$. Since the last three have the same form, due to the symmetry of the diagram in question, we will consider only one of them; the others can be obtained changing masses and exponents of propagators.

We quote only the results, where the analytic continuation process has been already carried out. The first triple series is,

$$\mathcal{M}_1(i,j,k,\{z\}) = \Big[f_1\mathcal{F}_C^{(3)}(-k,1-k-D/2;1+i+D/2,1+j+D/2,1-k-D/2)+(j\leftrightarrow k)+(i\leftrightarrow k)\Big]$$
$$+\Big[f_2\mathcal{F}_C^{(3)}(-j-k-D/2,1-j-k-D;1+i+D/2,1-j-D/2,1-k-D/2)+(i\leftrightarrow j)+(k\leftrightarrow i)\Big]$$
$$+f_3\mathcal{F}_C^{(3)}(-\sigma,1-\sigma-D/2;1-i-D/2,1-j-D/2,1-k-D/2), \tag{24}$$

where

$$f_1=(-\pi)^D(m_1^2)^{i+D/2}(m_2^2)^{j+D/2}(p^2)^k(-i|-D/2)(-j|-D/2),$$

$$f_2=(-\pi)^D(m_1^2)^{i+D/2}(-p^2)^{j+k+D/2}(-i|-D/2)(-j|-k-D/2)(j+k+D|-k-D/2)(-k|2k+D/2),$$

$$f_3=(-\pi)^D(p^2)^\sigma(-i|2i+D/2)(-j|2j+D/2)(-k|2k+D/2)(D/2|-\sigma-D/2).$$

Note that one of the solutions has a factor $(0|D/2)=\Gamma(D/2)/\Gamma(0)$, so that we have in fact seven terms (in positive dimension) instead of eight (in negative dimension.) The second triple series of hypergeometric type is represented by a sum of four terms,

$$\begin{aligned}\mathcal{M}_2(i,j,k,\{w\}) = {} & g_1\mathcal{F}_C^{(3)}(-k,D/2;1+i+D/2,1+j+D/2,D/2)\\ & +g_2\mathcal{F}_C^{(3)}(-j-k-D/2,-j;1-j-D/2,D/2,1+i+D/2)\\ & +g_3\mathcal{F}_C^{(3)}(-i,-i-k-D/2;1-i-D/2,1+j+D/2,D/2)\\ & +g_4\mathcal{F}_C^{(3)}(-\sigma,-i-j-D/2;1-i-D/2,1-j-D/2,D/2),\end{aligned} \tag{25}$$

where the hypergeometric series which appears in the above results

$$\mathcal{F}_C^{(3)}(\alpha,\beta;\gamma,\theta,\phi,\{x\}) = \sum_{n_1,n_2,n_3=0}^{\infty}\frac{(\alpha|n_{123})(\beta|n_{123})x_1^{n_1}x_2^{n_2}x_3^{n_3}}{n_1!n_2!n_3!(\gamma|n_1)(\theta|n_2)(\phi|n_3)}, \tag{26}$$

is a Lauricella function [14] which converges if,

$$|x_i|<1, \qquad \text{and} \qquad \sqrt{|x_1|}+\sqrt{|x_2|}+\sqrt{|x_3|}<1, \tag{27}$$

and we define,

$$g_1=\frac{(-\pi)^D}{(-1)^{i+j+k}}(m_1^2)^{i+D/2}(m_2^2)^{j+D/2}(m_3^2)^k(-i|-D/2)(-j|-D/2),$$

$$g_2=\frac{(-\pi)^D}{(-1)^{i+j+k}}(m_1^2)^{i+D/2}(m_3^2)^{j+k+D/2}(D/2|j)(-i|-D/2)(-k|-j-D/2),$$

$$g_3=\frac{(-\pi)^D}{(-1)^{i+j+k}}(m_2^2)^{j+D/2}(m_3^2)^{i+k+D/2}(-j|-D/2)(-k|-i-D/2)(D/2|i),$$

$$g_4=\pi^D(-m_3^2)^\sigma(-i|i+j+D/2)(-j|i+j+D/2)(D/2|-i-j-D)(-k|-i-j-D).$$

The variables in $\mathcal{M}_1$ and $\mathcal{M}_2$ are respectively $\{x\}=\{z\}$ and $\{x\}=\{w\}$, where

$$z_1=m_1^2/p^2, \qquad z_2=m_2^2/p^2, \qquad z_3=m_3^2/p^2,$$

$$w_1=m_1^2/m_3^2, \qquad w_2=m_2^2/m_3^2, \qquad w_3=p^2/m_3^2.$$

Observe that our results, which were obtained simultaneously, agree with Berends' [14] *et al* ones known in the literature (which was obtained by analytic continuation.) In their work, $\mathcal{M}_2(i,j,k,\{w\})$ was calculated first and then analytically continued to allow other values of momenta and masses, resulting in $\mathcal{M}_1(i,j,k,\{z\})$. However, if the analytic continuation formula was not known the result in the other region would be difficult to obtain. On the other hand, NDIM provides *all* the results simultaneously, allowing us to obtain also new analytic continuation formulas [8,11].

## V. INSERTIONS.

A third question the reader could pose is: And for higher number of loops does NDIM accomplish also good results? Our main objective is to apply NDIM to higher loops, but up to now we are able to "glue" diagrams with two external legs only.

Consider the diagram of figure 2. It has 4-loops and is represented by,

$$\mathcal{C}(i,j,k,m,n,s) = \int d^Dq\; d^Dr\; d^Dl\; d^Dt\;\; (q^2)^i (r^2)^j (p-q-r)^{2k} (l^2)^m (t^2)^n (r-l-t)^{2s}, \tag{28}$$

in a massless theory. Observe that the integrals in $l$ and $t$ are equal, see eq.(6), to $\mathcal{A}(m,n,s;p^2 \to r^2)$. Rewriting we get,

$$\begin{aligned}\mathcal{C}(i,j,k,m,n,s) &= \int d^Dq\; d^Dr\; (q^2)^i (r^2)^j (p-q-r)^{2k} \mathcal{A}(m,n,s;p^2 \to r^2), \quad (29)\\ &= \pi^D (-m|m+n+D/2)(-n|n+s+D/2)(-s|m+s+D/2)(\sigma_1+D/2|-2\sigma_1-D/2)\mathcal{A}(i,j+\sigma_1,k),\\ &= \pi^{2D}(p^2)^{\sigma_1+\sigma_2}(-i|i+j+\sigma_1+D/2)(-j-\sigma_1|j+\sigma_1+k+D/2)(-k|i+k+D/2)\\ &\quad\times(\sigma_2+D/2|-2\sigma_2-D/2)(-m|m+n+D/2)(-n|n+s+D/2)(-s|m+s+D/2)\\ &\quad\times(\sigma_1+D/2|-2\sigma_1-D/2), \end{aligned} \tag{30}$$

where $\sigma_1 = m+n+s+D$, $\sigma_2 = i+j+k+D$.

Let us study the diagram of figure 3, namely the one that has a single propagator replaced $N$-times by the graph of figure 1.

$$\mathcal{C}_N(\nu_1,\nu_2,...,\nu_{3N}) = \int \ldots \int \prod_{i=1}^{i=N} d^Dq_i\; d^Dr_i\; (q_i^2)^{\nu_{3i-2}} (r_i^2)^{\nu_{3i-1}} \left[(r_{i-1}-q_i-r_i)^2\right]^{\nu_{3i}}, \tag{31}$$

where $r_0^\mu = p^\mu$ is the external momentum and $N$ is the number of insertions. When $N=1$ the above integral reduces to $\mathcal{A}(\nu_1,\nu_2,\nu_3)$.

Applying the above procedure we can easily solve such scalar integral,

$$\mathcal{C}_N^{AC}(\nu_1,\nu_2,...,\nu_{3N}) = \mathcal{A}^{AC}(\nu_1,\nu_2,\nu_3)\mathcal{A}^{AC}(\nu_4,\nu_5+\sigma_1,\nu_6)\times ...\times \mathcal{A}^{AC}(\nu_{3N-2},\nu_{3N-1}+\sigma_1+...+\sigma_{N-1},\nu_{3N}), \tag{32}$$

where $\sigma_N = \nu_{3N-2}+\nu_{3N-1}+\nu_{3N}+D$.

*Water melon diagram. Massless case.*

Recently, the water melon diagram was considered in massive case, whether in four dimensions and analytic result (see first reference in [14]) or integral representations of it suitable for numerical calculations (see third reference in [14]).

Applying NDIM we can solve, exactly, such water melon diagrams, see figure 4. Here, we will consider the scalar massless case. At two-loop level the referred water melon is the graph of figure 1, our setting Sun diagram. At three-loop level we begin with,

$$G_{WM} = \int d^Dq\; d^Dr\; d^Dk\;\; \exp\left[-\alpha q^2 - \beta r^2 - \gamma k^2 - \theta(p-q-r-k)^2\right], \tag{33}$$

$$= \left(\frac{\pi^3}{\zeta}\right)^{D/2} \exp\left(-\frac{\alpha\beta\gamma\theta}{\zeta}p^2\right), \tag{34}$$

where $\zeta = \alpha\beta\gamma+\alpha\beta\theta+\alpha\gamma\theta+\beta\gamma\theta$. After a little bit of algebra we get a $4\times 4$ system which gives

$$\mathcal{W}_3(i,j,l,m) = \int d^Dq\; d^Dr\; d^Dk\; (q^2)^i (r^2)^j (k^2)^l (p-q-r-k)^{2m}, \tag{35}$$

$$= \frac{\pi^{3D/2} i!j!l!m!\Gamma(1-\rho_3-D/2)(p^2)^{\rho_3}}{(-1)^{i+j+l+m}\Gamma(1-i-D/2)\Gamma(1-j-D/2)\Gamma(1-l-D/2)\Gamma(1-m-D/2)\Gamma(1+\rho_3)}, \tag{36}$$

in negative dimension and Euclidean space. We define $\rho_3 = i+j+l+m+3D/2$. Generalizing this result to $N$-loops is quite easy,

$$\mathcal{W}_N(\{\nu_n\}) = \int \ldots \int \prod_{i=1}^{i=N} d^Dq_i\; (q_i^2)^{\nu_i} (p-q_1-q_2-\ldots-q_N)^{2\nu_{N+1}}, \tag{37}$$

$$= \frac{\pi^{ND/2}\nu_1!\ldots\nu_{N+1}!\Gamma(1-\rho_N-D/2)(p^2)^{\rho_N}}{(-1)^{\Sigma\nu}\Gamma(1-\nu_1-D/2)\ldots\Gamma(1-\nu_{N+1}-D/2)\Gamma(1+\rho_N)}, \tag{38}$$

where $\Sigma\nu = \nu_1+\ldots+\nu_{N+1}$ and $\rho_N = \Sigma\nu + ND/2$. Carrying the analytic continuation out we get,

$$\mathcal{W}_N^{AC}(\{\nu_n\}) = \pi^{ND/2}(p^2)^{\rho_N}(-\nu_1|2\nu_1+D/2)(-\nu_2|2\nu_2+D/2)\ldots(-\nu_N|2\nu_N+D/2)(\rho_N+D/2|-2\rho_N-D/2), \tag{39}$$

the result for negative exponents of propagators and positive dimension. As far as we know, this result was not known for arbitrary exponents of propagators. Observe that when one is allowed to "gluing" diagrams, the result can easily be generalized to N-loops.

## VI. CONCLUSION.

NDIM is a suitable technique to tackle the task of calculating multiloop Feynman integrals. Massless, massive, scalar, tensorial, even the ones for non-covariant gauges, such as the light-cone gauge [8,9] are easily performed. In all of them exponents of propagators are left arbitrary as well as the dimension $D$, just like in plain dimensional regularization. Usual parametric integrals are replaced by one Gaussian integral over each momentum flowing in the loop, and the main task is to solve systems of linear algebraic equations. Another point that we would like to stress is that no numerical calculations are required at all.


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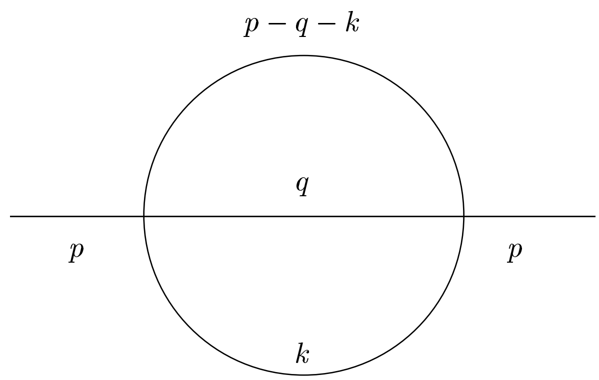


Figure 1: Simplest two-loop massless Feynman diagram, setting Sun.

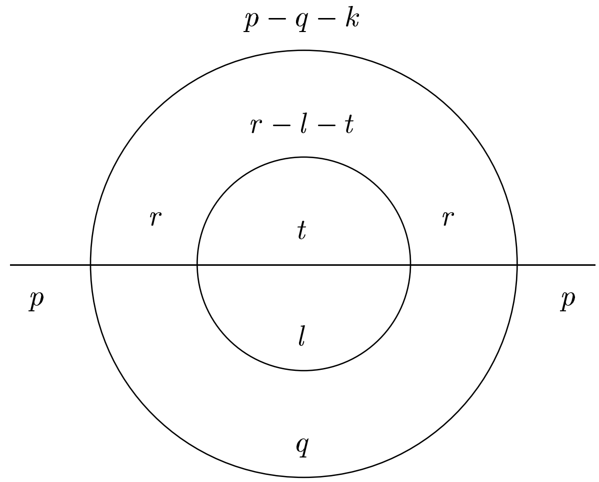


Figure 2: Four-loop massless Feynman diagram, setting Sun with one insertion.

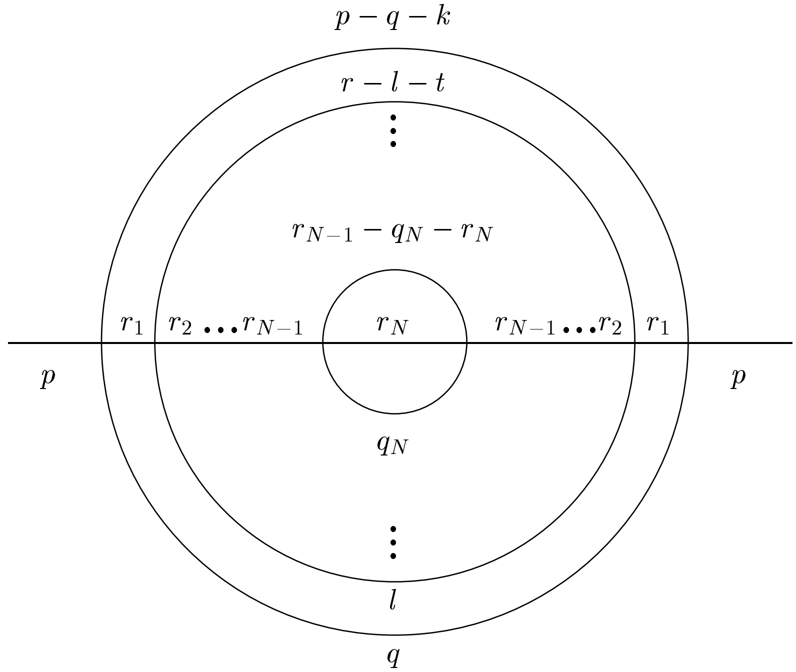


Figure 3: 2N-loop massless Feynman diagram, setting Sun with N insertions of the setting Sun graph.

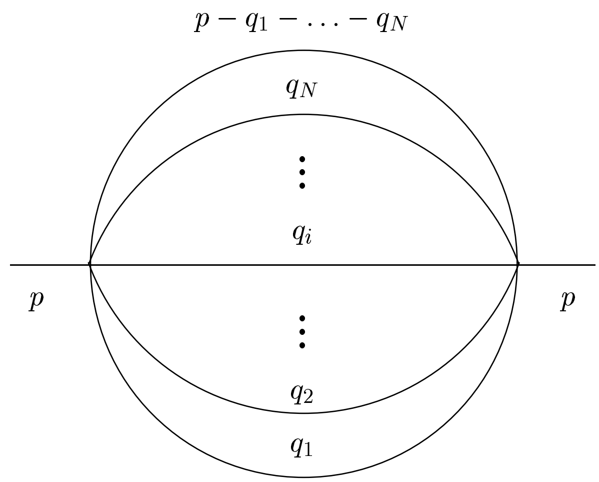


Figure 4: Scalar massless water melon Feynman diagram with $N$-loops.